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LETTER TO THE EDITOR

Wavelength-independent fringe spacing in rainbows from falling neutrons

M V Berry
H H Wills Physics Laboratory, Tyndall Avenue, Bristol, BS8 1TL, England

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Abstract. For particles with speed v and mass m emitted isotropically from a point source and falling under gravity g, the quantal probability density is dominated by a paraboloidal caustic decorated with paraboloidal interference fringes. Near the caustic, the fringes have a spacing independent of v and hence of wavelength, given by $\Delta R = 3.53897 \times (\hbar^2/m^2 g)^{1/3}$ for the first two (brightest) fringes at the level of the source. For neutrons in the Earth's field, $\Delta R = 0.02617$ mm. The effect might be difficult to detect.

Consider non-relativistic particles of mass m (hereinafter called neutrons) being sprayed isotropically with speed v from a point source and falling under a gravitational acceleration g. Each neutron moves in a parabolic orbit, and the whole family of orbits is enveloped by a caustic which is the 'bounding paraboloid' of elementary ballistics (figure 1). The radius of the paraboloid at the level of the source is v^2/g,

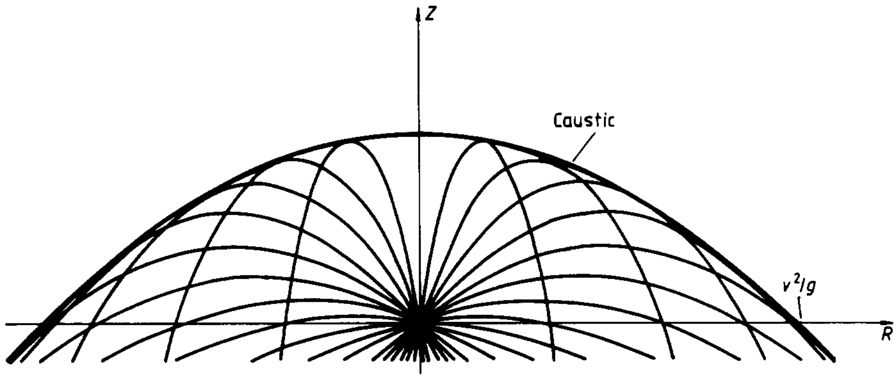

Figure 1. 'Gravity's rainbow' enveloping the orbits of falling particles emitted by a source.

and the dependence on v has been ingeniously exploited by Steyerl (1978) who constructed a 'gravity focusing spectrometer' to measure the velocity dispersion of ultracold neutron sources. This system is precisely analogous to the optical rainbow, whose principal features are focusing onto caustic cones in sunlight reflected and refracted by raindrops (Descartes 1637) and dispersion causing cones to appear in different directions for different colours (Newton 1730).

Each point within the caustic is reached by two orbits (figure 1) whose quantal interference determines the probability density $|\psi|^2$ for finding neutrons. The curious result reported here is that close to the caustic the spacing ΔR of the interference fringes is independent of v and hence of the neutron de Broglie wavelength. This follows from an elementary argument based on dimensional analysis, assuming ΔR to depend only on $\hbar$, m, g and v. The crucial fact is that the wave near the caustic is a fold diffraction catastrophe (Berry and Upstill 1980, Berry 1981), so that ΔR is proportional to $\hbar^{2/3}$. It then follows that

$$\Delta = \gamma(\hbar^2/m^2 g)^{1/3} \tag{1}$$

(where γ is a dimensionless number), and this indeed does not depend on v. Thus a manifestation of the wave nature of matter can be independent of the de Broglie wavelength. This is not a paradox, because ΔR does vanish in the classical limit $\hbar \to 0$.

A full analysis, leading to a determination of γ, proceeds by noting that the neutron wavefunction ψ is proportional to the time-independent Green function for neutrons with energy $\frac{1}{2}mv^2$, and this can be written as the Fourier transform of the propagator which for the present case of a uniform field is exactly equal to its semiclassical approximation (Jones and Papadopoulos 1971). Taking coordinates centred on the source with Z measured vertically upward and R perpendicular to the Z axis we obtain, for the case where N neutrons are emitted per unit time,

$$\psi(R, Z) = \frac{\hbar}{m}\left(\frac{\pi N}{v}\right)^{1/2} \int_0^\infty \mathrm{d}t \exp\{(\mathrm{i}/\hbar)[\tfrac{1}{2}mv^2 + W(R, Z, t)]\}$$
$$\times \frac{1}{(2\pi\hbar)^{3/2}}\left[\det\left(\frac{\partial^2 W}{\partial x_i\, \partial x_j}(R, Z, t)\right)\right]^{1/2}. \tag{2}$$

In this equation x_i denotes any spatial coordinate and W denotes Hamilton's principal function, given by a solution of

$$\frac{1}{2m}|\nabla W|^2 + mgZ = -\frac{\partial W}{\partial t} \tag{3}$$

namely

$$W(R, Z, t) = (m/2t)(R^2 + Z^2) - \tfrac{1}{2}mgZt - \tfrac{1}{24}mg^2t^3. \tag{4}$$

Henceforth only observations in the plane of the source ($Z = 0$) will be considered, so that

$$\psi(R, 0) = \left(\frac{mN}{2\hbar v}\right)^{1/2} \frac{1}{2\pi}\int_0^\infty \frac{\mathrm{d}t}{t^{3/2}} \exp\left\{\frac{\mathrm{i}m}{2\hbar}\left(v^2 t + \frac{R^2}{t} - \frac{g^2t^3}{12}\right)\right\}. \tag{5}$$

To evaluate this integral semiclassically, note that the integrand is dominated by its stationary points (where the derivative of phase with respect to t vanishes). Within the caustic ($R < v^2/g$) there are two such points on the real positive t axis; outside, there are none. On the caustic at $r = v^2/g$ the stationary points coincide at $t = (2v^2/g^2)^{1/2}$. Expanding the integrand to lowest non-trivial order about these R and t values gives the expectation value for the neutron density as

$$|\psi(R, 0)|^2 = N\left(\frac{m}{\hbar}\right)^{1/3} \frac{g^{5/3}}{2^{11/6}v^4} \mathrm{Ai}^2\left\{\left(R - \frac{v^2}{g}\right)\left(\frac{m^2 g}{4\hbar^2}\right)^{1/3}\right\} \tag{6}$$

where Ai denotes the function (Abramowitz and Stegun 1964) introduced by Airy (1838) precisely to describe waves near caustics. The Airy function decays exponentially for positive argument (outside the caustic) and oscillates for negative argument (within the caustic). The first bright fringes (maxima of Ai^2) correspond to arguments 1.01879 and 3.24820 and so, from (6), their spacing ΔR is

$$\Delta R = 3.53897(\hbar^2/m^2 g)^{1/3} \tag{7}$$

as expected on the basis of the dimensional reasoning leading to (1). A similar calculation shows that directly above the source the same fringes (paraboloids lying just within the classical caustic) have spacing $\Delta R/2$.

The independence of ΔR from v or the wavelength is a consequence not only of the gravitational force law but also of the dispersion relation for non-relativistic particles (it does not hold for falling light). As v increases, the radius of the caustic increases, and relative to this the fringes do get smaller as the pattern (6) moves rigidly outwards.

If the effect predicted here could be observed it would join a growing list of experiments (Bonse and Rauch 1979) confirming that quantal particles display interference and diffraction just as classical waves do, and setting upper limits on the magnitudes of possible nonlinear terms modifying Schrödinger's equation. Such an observation is likely to be difficult as the following considerations suggest. For neutrons in the Earth's gravity, (7) gives

$$\Delta R_{\mathrm{neutron}} = 0.02617 \text{ mm}. \tag{8}$$

This places stringent restrictions on the source. Firstly, its dimensions (and also that of the source) should be much smaller than ΔR. And secondly, to prevent the 'rainbow colours' from blurring the fringes the velocity dispersion $\Delta v/v$ should satisfy

$$\Delta v/v \ll g\Delta R/2v^2. \tag{9}$$

To keep the apparatus within laboratory bounds, the caustic radius should be less than about 10 m, i.e. $v < 10 \text{ ms}^{-1}$ (corresponding to a neutron temperature $T \sim 4$ mK), from which follows $\Delta v/v \ll 10^{-6}$. For electrons,

$$\Delta R_{\mathrm{electron}} = 3.927 \text{ mm} \tag{10}$$

which would be easier to detect were it not for the fact that the temperature would have to be $T \sim 2\ \mu$K for the electrons to fall within the laboratory; and there is the additional problem of preventing stray fields disturbing such slowly moving charges.

In principle these gravitational rainbows provide a quantum mechanical way of measuring g without knowing the velocity of the particles, but such an exotic technique is unlikely to replace conventional gravimetry.

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